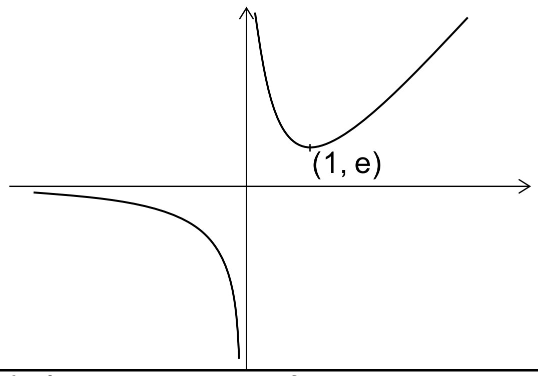
STEP MATHEMATICS 2

2022

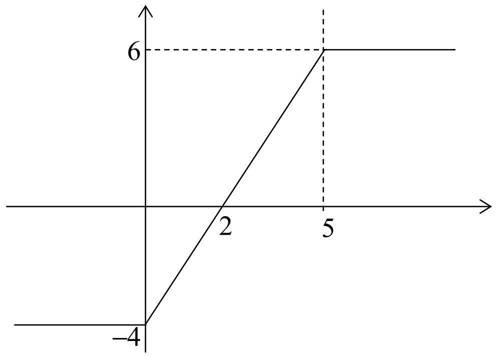
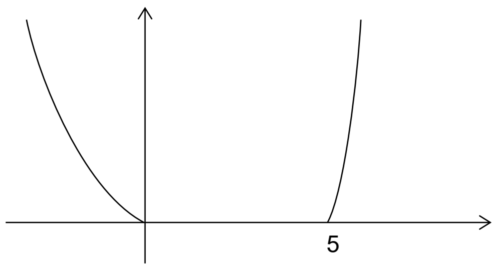
Mark Scheme

Question			Answer	Mark
1	(i)		$\int \frac{3x^3}{\sqrt{1+x^3}}\,dx = u \cdot v - \int u'v\,dx$	M1
			$\int \frac{3x^3}{\sqrt{1+x^3}}\,dx = x \cdot k\sqrt{1+x^3} - \int k\sqrt{1+x^3}\,dx$	M1
			$\int \frac{3x^3}{\sqrt{1+x^3}}\,dx = x \cdot 2\sqrt{1+x^3} - \int 2\sqrt{1+x^3}\,dx$	A1
			so $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\,dx = 2x\sqrt{1+x^3} + c$	A1
				[4]
	(ii)		$\frac{(x^2+2)\sin x}{x^3} = \frac{\sin x}{x} + \frac{2\sin x}{x^3}$	M1
			$\int \frac{2\sin x}{x^3}\,dx = -\frac{p}{x^2} \cdot \sin x + \int \frac{q\cos x}{x^2}\,dx$	M1
			$= -\frac{p}{x^2} \cdot \sin x - \frac{r}{x} \cdot \cos x - \int \frac{s\sin x}{x}\,dx$	M1
			$-\frac{1}{x^2} \cdot \sin x - \frac{1}{x} \cdot \cos x - \int \frac{\sin x}{x}\,dx$	
			so $\int \left(x^2+2\right)\frac{\sin x}{x^3}\,dx = -\frac{\sin x + x\cos x}{x^2} + c$	A1
				[4]
	(iii)	(a)	$\frac{dy}{dx} = \frac{(x-1)e^x}{x^2}$	M1
			Therefore there is a stationary point at (1, e).	A1
				
			Vertical asymptote at $x = 0$	G1
			Minimum in first quadrant and correct behaviour as $x \to \infty$	G1
			Correct behaviour as $x \to -\infty$	G1
				[5]

		(b)	$\int_a^{2a} \frac{e^x}{x^2}\,dx = \left[-\frac{p}{x}\cdot e^x\right]_a^{2a} + \int_a^{2a} \frac{qe^x}{x}\,dx$	**M1**
			$\int_a^{2a} \frac{e^x}{x^2}\,dx = \left[-\frac{1}{x}\cdot e^x\right]_a^{2a} + \int_a^{2a} \frac{e^x}{x}\,dx$	**A1**
			Therefore for integrals to be equal we need $\left[-\frac{1}{x}\cdot e^x\right]_a^{2a} = 0$	**M1**
			$-\frac{1}{2a}\cdot e^{2a} + \frac{1}{a}\cdot e^a = 0$ $\frac{1}{2a}\cdot e^a\left(-e^a + 2\right) = 0$	**M1**
			so $a = \ln 2$	**A1**
				[5]
		(c)	As before, this means we would need $\left[-\frac{1}{x}\cdot e^x\right]_m^n$ i.e. $\frac{e^n}{n} = \frac{e^m}{m}$	**B1**
			From the graph in part **(iii) (a)** this would mean that the smaller of n, m must lie in the range (0, 1). Hence this is not an integer.	**E1**
				[2]

Question			Answer	Mark
2	(i)		$u_{n+2} - u_{n+1} = u_{n+1} - u_n$	M1
			so constant differences.	A1
			If $u_n - u_{n-1} = d$, then $u_n = u_1 + (n-1)d$ which is of degree at most 1	B1
				[3]
	(ii)		$t_{n+1} + p(n+1)^2$ $= \frac{1}{2}(t_{n+2} + p(n+2)^2 + t_n + pn^2) - p$	M1
			so $t_{n+1} = \frac{1}{2}(t_{n+2} + t_n)$	A1
			so t_n has degree at most 1	A1
			Hence since $p \neq 0$, v_n has degree 2.	A1
			Taking $v_n = pn^2 + qn + r$, gives: $p + q + r = 0$ $4p + 2q + r = 0$	M1
			so $q = -3p$	A1
			And $r = 2p$	A1
				[7]
	(iii)		Substitutes $w_n = t_n + kn^3$, so	B1
			$t_{n+1} + k(n+1)^3$ $= \frac{1}{2}(t_{n+2} + k(n+2)^3 - t_n + kn^3) - an - b$	M1
			LHS and RHS both give $kn^3 + 3kn^2$ terms	A1
			$$t_{n+1} = \frac{1}{2}(t_{n+2} + t_n) + (3k - a)n - (b - 3k)$$	A1
			Choosing $k = \frac{1}{3}a$	A1
			gives case **(ii)** (with $p = b - a$) so t_n has degree at most 2 and w_n has degree 3, as $a \neq 0$.	A1
			unless $b = a$, when case **(i)** applies so t_n has degree at most 2 and w_n has degree 3, as $a \neq 0$.	A1
			Taking $w_n = \frac{1}{3}an^3 + (b-a)n^2 + qn + r$ gives $b - \frac{2}{3}a + q + r = 0$ $-\frac{4}{3}a + 4b + 2q + r = 0$	M1
			so $q = \frac{2}{3}a - 3b$	A1
			and $r = 2b$	A1
			$w_n = \frac{1}{3}an^3 + (b-a)n^2 + \left(\frac{2}{3}a - 3b\right)n + 2b$	
				[10]

Question			Answer	Mark
3	(i)		Base case: $F_n \le 2^{n-n}F_n$	B1
			For $n \ge 1$, $F_{n-1} \le F_n$, so if $r \ge n$ and $F_r \le 2^{r-n}F_n$	M1
			$F_{r+1} \le 2F_r \le 2^{(r+1)-n}F_n$	A1
			Logical structure correct, with conclusion.	A1
				[4]
	(ii)		$\sum_{r=1}^{n}\frac{F_{r+1}}{10^{r-1}}-\sum_{r=1}^{n}\frac{F_r}{10^{r-1}}-\sum_{r=1}^{n}\frac{F_{r-1}}{10^{r-1}}$ $=100\sum_{r=1}^{n}\frac{F_{r+1}}{10^{r+1}}-10\sum_{r=1}^{n}\frac{F_r}{10^r}-\sum_{r=1}^{n}\frac{F_{r-1}}{10^{r-1}}$	M1
			$100\sum_{r=2}^{n+1}\frac{F_r}{10^r}-10\sum_{r=1}^{n}\frac{F_r}{10^r}-\sum_{r=0}^{n-1}\frac{F_r}{10^r}$	M1
			$=100(S_n+\cdots)-10S_n-(S_n+\cdots)$	A1
			$=100\left(\ldots+\frac{F_{n+1}}{10^{n+1}}-\frac{F_1}{10}\right)-\left(\ldots+F_0-\frac{F_n}{10^n}\right)$	A1
				[4]
	(iii)		In **(ii)**, the left hand side is equal to zero, so $89S_n = 10F_1+F_0-\frac{F_n}{10^n}-\frac{F_{n+1}}{10^{n-1}}$	M1
			but $\frac{F_n}{10^n}+\frac{F_{n+1}}{10^{n-1}} \to 0$ as $n \to \infty$, from **(i)**	B1
			and $F_0 = 0$, so $S_\infty = \frac{10}{89}$	A1
			$\sum_{r=7}^{\infty}\frac{F_r}{10^r} \leqslant \frac{F_7}{10^7}\sum_{r=0}^{\infty}\frac{2^r}{10^r} = \frac{13}{10^7\left(1-\frac{2}{10}\right)} < 2\times10^{-6}$	M1
			$\frac{1}{89}=\frac{1}{10}\left(\frac{1}{10}+\frac{1}{10^2}+\frac{2}{10^3}+\frac{3}{10^4}+\frac{5}{10^5}+\frac{8}{10^6}+\sum_{r=7}^{\infty}\frac{F_r}{10^r}\right)$ $=0.0112358+\varepsilon$	A1
			with $\varepsilon < 2\times10^{-7}$, so the first six digits of the decimal expansion of $\frac{1}{89}$ are 0·011235	A1
				[6]
	(iv)		Let $T_n = \sum_{r=1}^{n}\frac{F_r}{100^r}$	M1
			then $0=\sum_{r=1}^{n}\frac{F_{r+1}}{100^{r-1}}-\sum_{r=1}^{n}\frac{F_r}{100^{r-1}}-\sum_{r=1}^{n}\frac{F_{r-1}}{100^{r-1}}$ $=10000\left(T_n+\frac{F_{n+1}}{100^{n+1}}-\frac{1}{100}\right)-100T_n-\left(T_n-\frac{F_n}{100^n}\right)$	M1
			$9899T_n = 100-\frac{F_n}{100^n}-\frac{F_{n+1}}{100^{n-1}}$	A1
			so $T_\infty = \frac{100}{9899}$ and $\frac{1}{9899}$ is the required fraction	A1
			as $\frac{1}{9899} = 0.0001010203050813213455+\varepsilon$	M1
			where $\varepsilon \leqslant \frac{89}{1-\frac{2}{100}}\times10^{-24} < 10^{-22}$	A1
				[6]

Question			Answer	Mark
4	(i)		$For\ x \le 0, \quad \lvert x\rvert = -x \quad \lvert x-5\rvert = -(x-5)$ $For\ 0 \le x \le 5 \quad \lvert x\rvert = x \quad \lvert x-5\rvert = -(x-5)$ $For\ 5 \le x \quad \lvert x\rvert = x \quad \lvert x-5\rvert = x-5$	M1
			$For\ x \le 0, \quad f(x) = -x - \big(-(x-5)\big) + 1 = -4$ $For\ 0 \le x \le 5 \quad f(x) = x - \big(-(x-5)\big) + 1 = 2x-4$ $For\ 5 \le x \quad f(x) = x-(x-5)+1 = 6$	A1
				G1 G1
				[4]
	(ii)		Writing $g(x) = a\lvert x\rvert + b\lvert x-5\rvert + c$	M1
			$For\ x \le 0, \quad g(x) = -ax + b\big(-(x-5)\big) + c$ $For\ 0 \le x \le 5 \quad g(x) = ax + b\big(-(x-5)\big) + c$ $For\ 5 \le x \quad g(x) = ax + b(x-5) + c$	M1
			Coefficients of x: $-a-b=-1$ $a-b=3$ $a+b=1$	
			$a=2,\ b=-1$ So $c=5$	
			$g(x) = 2\lvert x\rvert - \lvert x-5\rvert + 5$	A1
				[3]
				
			Convex quadratic shapes of appropriate gradient and without vertex in $(-\infty,0]$, $[5,\infty)$	G1
			Horizontal section in $[0,5]$, with discontinuous gradient at endpoints.	G1
			Appropriate asymmetry of quadratic parts	G1
				[5]
	(iv)		$k(x) = x^2 - \lvert x(x-5)\rvert +$ linear, constant terms	M1
			$k(x) - x^2 + \lvert x(x-5)\rvert$ is:	M1
			$x \leqslant 0$: $10x - x^2 + x(x-5) = 5x$ $0 \leqslant x \leqslant 5$: $2x^2 - x^2 - x(x-5) = 5x$ $5 \leqslant x$: $50 - x^2 + x(x-5) = 50-5x$	A1
			Set equal to $a + b\lvert x-5\rvert$	M1

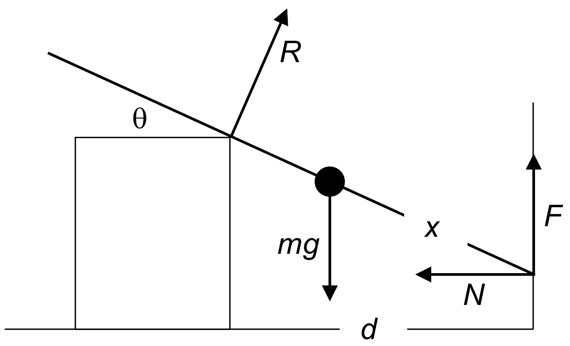
			Determine by substitution necessary values of a and b	**M1**
			$a = 25$ and $b = -5$	**A1**
			Verification that these are sufficient	**A1**
			Thus $k(x) = x^2 - \lvert x(x-5)\rvert + 25 - 5\lvert x-5\rvert$	**A1**
				[8]

Question			Answer	Mark
5	(i)		As z, y non-negative and $a > b, c$: $ay \geqslant by$ and $az \geqslant cz$	B1
				[1]
	(ii)	(a)	$\Delta = \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz$	B1
		(b)	By **(i)**, $(x + y + z) \geqslant \frac{2\Delta}{a}$	M1
			$\frac{2\Delta}{a}$ is the minimum value	A1
			[as this lower bound is attained at] $\left(\frac{2\Delta}{a},\ 0,\ 0\right)$.	A1
				[4]
	(iii)	(a)	Correct number of terms for expansions of any two of: $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ $(bx - ay)^2 + (cy - bz)^2 + (az - cx)^2$ $(ax + by + cz)^2$	M1
			Fully correct expansions.	A1
			Given result fully shown.	A1
				[3]
	(iii)	(b)	By **(iii)**, $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ $= (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 + (2\Delta)^2$	M1
			so the minimum value of $x^2 + y^2 + z^2$ is $\frac{4\Delta^2}{a^2+b^2+c^2}$	A1
			This occurs when $bx = ay$, $cy = bz$ and $az = cx$	M1
			so when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \lambda$, say, where $\lambda > 0$.	M1
			Then $\Delta = \frac{1}{2}a(a\lambda) + \frac{1}{2}b(b\lambda) + \frac{1}{2}c(b\lambda)$	M1
			so $\lambda = \frac{2\Delta}{a^2+b^2+c^2}$	A1
			minimum at $(a\lambda, b\lambda, c\lambda)$ with this value of λ.	A1
				[7]
	(iv)		$(ax + by + cz)^2 \geq (cx + cy + cz)^2$	M1
			$= c^2(x + y + z)^2 \geq c^2(x^2 + y^2 + z^2)$	M1
			so $x^2 + y^2 + z^2 \leqslant \frac{4\Delta^2}{c^2}$	M1
			Maximum of $\frac{4\Delta^2}{c^2}$	A1
			at $\left(0,\ 0,\ \frac{2\Delta}{c}\right)$.	A1
				[5]

Question			Answer	Mark
6	(i)	(a)	Differentiating implicitly with respect to x gives $2x + 2y\frac{dy}{dx} = 2a$ so, by substitution, $x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$	B1
			For second family: $2x + 2y\frac{dy}{dx} = 2b\frac{dy}{dx}$	M1
			so $y\left(2x + 2y\frac{dy}{dx}\right) = (x^2 + y^2)\frac{dy}{dx}$	A1
			$(x^2 - y^2)\frac{dy}{dx} = 2xy$	A1
				[4]
		(b)	The product of the gradients at points (x, y) where the curves meet is $\frac{y^2 - x^2}{2xy} \times \frac{2xy}{x^2 - y^2} = -1$, provided $x \neq y$, So the tangents to the curves at these points are perpendicular	B1
			At (c, c), for the first family of curves: $$2c^2\frac{dy}{dx} = 0$$ and so $\frac{dy}{dx} = 0$. For the second family of curves: $$2c^2\frac{dx}{dy} = 0$$ and so $\frac{dy}{dx} = \infty$.	M1
			and the tangents to the circles $(x - c)^2 + y^2 = c^2$ and $(y - c)^2 + x^2 = c^2$ at this point are $y = c$ and $x = c$.	A1
			which are indeed perpendicular.	A1
				[4]
	(ii)		First family $\frac{dy}{dx} = \frac{c}{x}$	M1
			so $x\ln x\frac{dy}{dx} = y$	A1
			so orthogonal family has $y\frac{dy}{dx} = -x\ln x$	A1
			solving differential equation by separating variables	M1
			$\int -x\ln x\,\mathrm{d}x = -\frac{1}{2}x^2\ln x - \int -\frac{1}{2}x^2 \cdot \frac{1}{x}\mathrm{d}x$	M1
			$= -\frac{1}{2}x^2\, ln\, x + \frac{1}{4}x^2$	A1
			so $\frac{1}{2}y^2 = -\frac{1}{2}x^2\, ln\, x + \frac{1}{4}x^2 + c$	A1
				[7]
	(iii)		If two curves, with parameters k_1, k_2 meet, require $4k_1(x + k_1) = 4k_2(x + k_2)$	M1
			so $x = -(k_1 + k_2)$	A1
			$y^2 = -4k_1k_2$	A1
			for any curve, $2y\frac{dy}{dx} = 4k$	M1
			so the gradients of the two curves satisfy $\left.\frac{dy}{dx}\right\vert_1 \cdot \left.\frac{dy}{dx}\right\vert_2 = \frac{2k_1}{y}\frac{2k_2}{y} = -1$	A1 CSO
				[5]

Question			Answer	Mark
7	(i)	(a)	$w^5 = -\frac{w+n}{nw+1}$	M1
			$\lvert w^5 \rvert = \left\lvert \frac{w+n}{nw+1} \right\rvert = \sqrt{\left(\frac{w+n}{nw+1}\right)\overline{\left(\frac{w+n}{nw+1}\right)}}$	M1
			$= \sqrt{\frac{w+n}{nw+1}\frac{\bar{w}+n}{n\bar{w}+1}} = \sqrt{\frac{w\bar{w}+n(w+\bar{w})+n^2}{n^2w\bar{w}+n(w+\bar{w})+1}}$	A1
			which gives the required result, as $w + \bar{w} = 2\,Re(w)$	A1
				[4]
		(b)	$f(w) - g(w) = (n^2 - 1)(1 - \lvert w \rvert^2)$	M1
			and $n > 1$, so if $\lvert w \rvert < 1$, $f(w) - g(w) > 0$	A1
			but since f(w) and g(w) are both positive (each is the square of the magnitude of a complex number) f(w) > g(w) > 0	A1
			so $\frac{f(w)}{g(w)} > 1$ and so $\lvert w \rvert = \sqrt[10]{\left\lvert \frac{f(w)}{g(w)} \right\rvert} > 1$ # Hence $\lvert w \rvert \geq 1$	A1
				[4]
		(c)	if $\lvert w \rvert > 1$, $f(w) - g(w) < 0$	M1
			so $\frac{f(w)}{g(w)} < 1$	A1
			so $\lvert w \rvert = \sqrt[10]{\left\lvert \frac{f(w)}{g(w)} \right\rvert} < 1$ #. Hence $\lvert w \rvert \leq 1$	A1
			and, in combination with **(b)**, this gives $\lvert w \rvert = 1$	A1
				[4]
	(ii)	(a)	Since the coefficients of h(z) are real, but none of the roots is purely real, the six roots occur in conjugate pairs	B1
			Suppose $p \pm iq$ are roots; then quadratic factor of $(z - p - iq)(z - p + iq) = (z^2 - 2pz + p^2 + q^2)$ with $2p$ real and $p^2 + q^2 = \lvert z \rvert^2 = 1$ by **(i)(c)**	M1
			Hence the algebraic factors are as stated, and the only remaining possibility is a numerical factor, which must be n by comparison of the z^6 term	A1
				[3]
		(b)	$a_1 + a_2 + a_3$ is the sum of all six roots, so equal to $-\frac{1}{n}$	B1
				[1]
		(c)	The coefficient of z^3 in h is $-a_1a_2a_3 - 2a_1 - 2a_2 - 2a_3$	M1
			which must be zero so $a_1a_2a_3 = \frac{2}{n}$	A1
				[2]
		(d)	The sum of a_1, a_2, a_3 is negative, so they cannot all be positive, but their product is positive, so exactly two of them are negative	B1
			hence exactly four roots of the equation have negative real part	B1
				[2]

Question			Answer	Mark
8	(i)		If neither parallel to the *y*-axis, their gradients satisfy $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ m \end{pmatrix}$ with $\lambda \neq 0$	M1
			eliminating λ from $c + dm = \lambda m$, $a + bm = \lambda$	M1
			$\Leftrightarrow c + dm = m(a + bm)$	A1
			If $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is invariant, then $b = 0$	M1
			and the gradient of the other line satisfies $(a - d)m = c$	A1
				[5]
	(ii)		If $b \neq 0$, and the angle θ between the lines is 45° then $\cos^2\theta = \frac{1}{2}$, so using the scalar product	B1
			$\left(\begin{pmatrix} 1 \\ m_1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ m_2 \end{pmatrix}\right)^2 = \frac{1}{2}(1 + m_1^2)(1 + m_2^2)$	B1
			so $(1 + m_1 m_2)^2 + 4m_1 m_2 = (m_1 + m_2)^2$	M1
			so $\left(1 - \frac{c}{b}\right)^2 - 4\frac{c}{b} = \frac{(a-d)^2}{b^2}$	A1
			If $b = 0$, the condition is $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} a-d \\ c \end{pmatrix}\right)^2 = \frac{1}{2}((a-d)^2 + c^2)$	M1
			so $c^2 = (a - d)^2$ as required	A1
				[6]
	(iii)		If $b \neq 0$, the angles with $y = x$ are equal iff $\begin{pmatrix} 1 \\ m_1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ m_2 \end{pmatrix}$ make equal angles with $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	B1
			so $\frac{\left(\begin{pmatrix} 1 \\ m_1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^2}{2(1+m_1^2)} = \frac{\left(\begin{pmatrix} 1 \\ m_2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^2}{2(1+m_2^2)}$	M1
			$(1 + m_2^2)(1 + m_1)^2 = (1 + m_1^2)(1 + m_2)^2$ so $(m_1 - m_2)(1 - m_1 m_2) = 0$	A1
			but $m_1 \neq m_2$ so requirement is $m_1 m_2 = 1$	B1
			which is $b + c = 0$	A1
			If $b = 0$, require $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ also invariant	M1
			so $c = 0$, which is the same condition	A1
				[7]
	(iv)		Require $c = -b$ and $(a - d)^2 = 8b^2$	M1
			so e.g. $\begin{pmatrix} 2\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} \sqrt{2} & 1 \\ -1 & -\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} \sqrt{2} & -2 \\ 2 & 5\sqrt{2} \end{pmatrix}$ etc	A1
				[2]

Question			Answer	Mark
9				
				
			Diagram: correct location of plank, prism, wall	G1
			and all forces	G1
			For equilibrium: $F + R\cos\theta - mg = 0$	B1
			and $R\sin\theta - N = 0$	B1
			and $R\cdot d\sec\theta = mgx\cos\theta$	B1
			so $R = \frac{mgx\cos^2\theta}{d}$, $N = \frac{mgx\sin\theta\cos^2\theta}{d}$	M1
			$F = mg\left(1 - \frac{x\cos^3\theta}{d}\right)$	A1
				[7]
	(i)		so if $x = d\sec^3\theta$, $F = 0$	B1
				[1]
	(ii)		If $x > d\sec^3\theta$, F is negative so necessary that $$mg\left(\frac{x\cos^3\theta}{d} - 1\right) \leqslant \mu\frac{mgx\sin\theta\cos^2\theta}{d}$$	M1
			$$\mu \geqslant \frac{x\cos^3\theta - d}{x\sin\theta\cos^2\theta}$$	A1
			If $x < d\sec^3\theta$, F is positive so necessary that $mg\left(1 - \frac{x\cos^3\theta}{d}\right) \leqslant \mu\frac{mgx\sin\theta\cos^2\theta}{d}$	M1
			so $\mu \geqslant \frac{d\sec^3\theta - x}{x\tan\theta}$	A1
				[4]
	(iii)		If $\frac{x}{d} \geqslant \sec^3\theta$ then $\frac{x}{d} \geqslant \frac{\sec^3\theta}{1+\mu\tan\theta}$	B1
			if $\frac{x}{d} < \sec^3\theta$ require $\mu \geqslant \frac{d\sec^3\theta - x}{x\tan\theta}$	M1
			so $\mu x\tan\theta + x \geqslant d\sec^3\theta$	A1
			When $\mu < \cot\theta$, if $\frac{x}{d} \leqslant \sec^3\theta$, then $\frac{x}{d} \leqslant \frac{\sec^3\theta}{1-\mu\tan\theta}$	B1
			if $\frac{x}{d} > \sec^3\theta$ require $\mu \geqslant \frac{x - d\sec^3\theta}{x\tan\theta}$	M1
			$d\sec^3\theta \geqslant x - \mu x\tan\theta$	A1
				[6]
	(iv)		Now require $x < d\sec\theta$, so $\sec\theta > \frac{\sec^3\theta}{1+\mu\tan\theta}$, by the first inequality in **(iii)**	M1
			so $\mu\tan\theta > \sec^2\theta - 1 = \tan^2\theta$	A1
				[2]

Question			Answer	Mark
10	(i)		$h = ut\sin\alpha - \frac{1}{2}gt^2$ and $s = ut\cos\alpha$	B1
			so $h = \frac{us}{u\cos\alpha}\sin\alpha - \frac{1}{2}g\frac{s^2}{u^2\cos^2\alpha}$ or $t = \frac{s}{u\cos\alpha}$	B1
				[2]
	(ii)		require $y\tan\theta = \tan\alpha\sqrt{x^2+y^2} - \frac{g}{2u^2}(x^2+y^2)(1+\tan^2\alpha)$	M1
			or real solutions to $a\tan^2\alpha - b\tan\alpha + c = 0$ with $a = \frac{g}{2u^2}(x^2+y^2)$, $b = \sqrt{x^2+y^2}$, $c = \frac{g}{2u^2}(x^2+y^2) + y\tan\theta$	A1
			so $(x^2+y^2) \geqslant 4\frac{g}{2u^2}(x^2+y^2)\left(\frac{g}{2u^2}(x^2+y^2) + y\tan\theta\right)$	M1
			so $\frac{u^4}{g^2} \geqslant x^2 + y^2 + \frac{2yu^2}{g}\tan\theta$	A1
			$$\frac{u^4}{g^2} + \frac{u^4}{g^2}\tan^2\theta \geqslant x^2 + \left(y + \frac{u^2}{g}\tan\theta\right)^2$$	A1
				[5]
	(iii)		If $x = 0$, the condition can be written as $\left(y + \frac{u^2\tan\theta}{g}\right)^2 \leqslant \frac{u^4}{g^2}\sec^2\theta \cdots$	M1
			$$-\frac{u^2}{g}\tan\theta \pm \frac{u^2}{g}\sec\theta$$	A1 A1
			distance up plane $d = y\sec\theta$ satisfies $d \leqslant \frac{u^2}{g}\sec\theta(\sec\theta - \tan\theta)$	M1
			so greatest d is $\frac{u^2}{g}\frac{1-\sin\theta}{\cos^2\theta} = \frac{u^2}{g}\frac{1-\sin\theta}{1-\sin^2\theta}$	A1
			also, $d \geqslant -\frac{u^2}{g}\frac{(1+\sin\theta)}{\cos^2\theta} = -\frac{u^2}{g(1-\sin\theta)}$ so greatest distance down slope is $\frac{u^2}{g(1-\sin\theta)}$	A1
				[6]
	(iv)		If $y = 0$, the condition can be written as $x^2 \leqslant \frac{u^4}{g^2}$	M1
			so the length of road is $\frac{2u^2}{g}$	A1
			If the gun is moved a distance r up the slope, the condition is derived by substituting $y - r\cos\theta$ for y	M1
			so $x^2 + \left(y - r\cos\theta + \frac{u^2\tan\theta}{g}\right)^2 \leqslant \frac{u^4}{g^2}(1+\tan^2\theta)$	A1
			so when $y = 0$, $x^2 \leqslant \frac{u^4}{g^2}(1+\tan^2\theta) - \left(\frac{u^2\tan\theta}{g} - r\cos\theta\right)^2$	M1
			which is maximised by $r = \frac{u^2}{g}\tan\theta\sec\theta$	A1
			with length of road reached $\frac{2u^2}{g}\sec\theta$	A1
				[7]

Question			Answer	Mark
11	(i)		Expected net loss is $q^T(\cdots)$	M1
			$(\cdots(1-q^{N-T})-\cdots q^{N-T})$	M1
			$=q^T(D(1-q^{N-T})-(N-T)q^{N-T})$	A1
				[3]
	(ii)		all variables non-negative and $N\geq T$, $D>0$, so denominator positive so $\alpha\geq 0$.	B1
			$N(N-T+D)-DT=(N+D)(N-T)>0$, so $\alpha<1$	B1
			$\frac{d}{dq}[expected\, net\, loss]=0$	M1
			$TDq^{T-1}-N(N-T+D)q^{N-1}=0$	A1
			$N(N-T+D)q^{T-1}(\alpha-q^{N-T})=0$	A1
			hence $q=\alpha^{\frac{1}{N-T}}$ determines exactly one value of q with $0\leqslant q<1$ for which the expected net loss is stationary	A1
			$\frac{d^2}{dq^2}[expected\, net\, loss]$ $=T(T-1)Dq^{T-2}-N(N-1)(N-T+D)q^{N-2}$	M1
			at the root $=N(N-T+D)q^{T-2}\big((T-1)\alpha-(N-1)q^{N-T}\big)$	M1
			$=N(N-T+D)q^{N-2}\big((T-1)-(N-1)\big)$ at the root	M1
			but $-N(N-T)(N-T+D)q^{N-2}<0$, so maximum	A1
			The maximum net loss is $q^T\big(D-\alpha(N-T+D)\big)$	M1
			$=\frac{Dq^T}{N}(N-T)$ but $q^T=(q^{N-T})^{\frac{T}{N-T}}=\alpha^k$	A1
				[12]
	(iii)		The expected loss is an increasing function of T if the expected net loss with one extra stick tested is larger than that without the extra stick	M1
			so when $[Dq^{T+1}-q^N(N-(T+1)+D)]-[Dq^T-q^N(N-T+D)]$ $=q^T(q^{N-T}-Dp)>0$ for all T	A1
			which is the case if $q^{N-T}>Dp$	A1
			As p tends to zero, the left hand side of this expression tends to 1, and the right hand side to 0 hence there exists $\beta>0$ such that, for all $p<\beta$, the expected net loss is an increasing function of T	E1
			Thus for $p<\beta$, testing no sticks minimises the expected net loss.	E1
				[5]

Question			Answer	Mark
12	(i)		$\int_0^1 kx^n(1-x)\,dx = \left[k\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right)\right]_0^1 = \frac{k}{(n+1)(n+2)}$	B1
			$\mu = \int_0^1 kx^{n+1}(1-x)\,dx = \left[k\left(\frac{x^{n+2}}{n+2}-\frac{x^{n+3}}{n+3}\right)\right]_0^1$	M1
			$=\frac{n+1}{n+3}$	A1
				[3]
	(ii)		μ less than the median if $\int_0^{\mu} kx^n(1-x)\,dx < \frac{1}{2}$	M1
			so if $k\left(\frac{\mu^{n+1}}{n+1}-\frac{\mu^{n+2}}{n+2}\right) < \frac{1}{2}$	A1
			$2\left((n+2)-\frac{(n+1)^2}{n+3}\right) < \left(\frac{n+3}{n+1}\right)^{n+1}$	A1
			$(n+2)-\frac{(n+1)^2}{n+3} = \frac{3n+5}{n+3} = 3-\frac{4}{n+3}$ and $\frac{n+3}{n+1} = 1+\frac{2}{n+1}$	A1
			[The terms of the expansion are all positive, so the inequality holds if] $1+(n+1)\frac{2}{n+1}+\frac{(n+1)n}{2}\left(\frac{2}{n+1}\right)^2$ $+\frac{(n+1)n(n-1)}{6}\left(\frac{2}{n+1}\right)^3 > 6-\frac{8}{n+3}$	M1
			expansion gives $1+2+\frac{2n}{n+1}+\frac{4n(n-1)}{3(n+1)^2}$	A1
			[so if] $6n(n+1)(n+3)+4n(n-1)(n+3)$ $>9(n+3)(n+1)^2-24(n+1)^2$	M1
			$2n(n+3)(5n+1) > 3(3n+1)(n^2+2n+1)$	A1
			$n^3+11n^2-9n-3>0$	A1
			[so if] $(n-1)(n^2+12n+3)>0$	A1
			which is certainly the case if $n > 1$	A1
				[11]
	(iii)		The mode m satisfies $f'(m) = k\left(nm^{n-1}-(n+1)m^n\right) = 0$	M1
			so $m = \frac{n}{n+1}$	A1
			$\int_0^m kx^n(1-x)\,dx = \left[k\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right)\right]_0^m$	M1
			$=2\left(\frac{n}{n+1}\right)^{n+1}$	A1
			but the given result implies	M1

			$\left(\frac{n+1}{n}\right)^{n+1} < \left(\frac{1+1}{1}\right)^{1+1} = 4$ for $n > 1$	
			so $\int_0^m kx^n(1-x)\,\mathrm{d}x > 2 \cdot \frac{1}{4} = \frac{1}{2}$ and hence the mode is greater than the median.	**A1**
				[6]